

Uncertainty

Let action A_t = leave for airport t minutes before flight
Will A_t get me there on time?

Problems:

- 1) partial observability (road state, other drivers' plans, etc.)
- 2) noisy sensors (KCBS traffic reports)
- 3) uncertainty in action outcomes (flat tire, etc.)
- 4) immense complexity of modelling and predicting traffic

Hence a purely logical approach either

- 1) risks falsehood: " A_{25} will get me there on time"
- or 2) leads to conclusions that are too weak for decision making:
" A_{25} will get me there on time if there's no accident on the bridge
and it doesn't rain and my tires remain intact etc etc."

(A_{1440} might reasonably be said to get me there on time
but I'd have to stay overnight in the airport ...)

Probability

Probabilistic assertions **summarize** effects of

laziness: failure to enumerate exceptions, qualifications, etc.

ignorance: lack of relevant facts, initial conditions, etc.

Subjective or Bayesian probability:

Probabilities relate propositions to one's own state of knowledge

e.g., $P(A_{25} | \text{no reported accidents}) = 0.06$

These are **not** claims of a “probabilistic tendency” in the current situation
(but might be learned from past experience of similar situations)

Probabilities of propositions change with new evidence:

e.g., $P(A_{25} | \text{no reported accidents, 5 a.m.}) = 0.15$

(Analogous to logical entailment status $KB \models \alpha$, not truth.)

Probability basics

Begin with a set Ω —the sample space

e.g., 6 possible rolls of a die.

$\omega \in \Omega$ is a sample point/possible world/atomic event

A probability space or probability model is a sample space with an assignment $P(\omega)$ for every $\omega \in \Omega$ s.t.

$$0 \leq P(\omega) \leq 1$$

$$\sum_{\omega} P(\omega) = 1$$

e.g., $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$.

An event A is any subset of Ω

$$P(A) = \sum_{\{\omega \in A\}} P(\omega)$$

E.g., $P(\text{die roll} < 4) = P(1) + P(2) + P(3) = 1/6 + 1/6 + 1/6 = 1/2$

Random variables

A **random variable** is a function from sample points to some range, e.g., the reals or Booleans

e.g., $Odd(1) = true$.

P induces a **probability distribution** for any r.v. X :

$$P(X = x_i) = \sum_{\{\omega: X(\omega) = x_i\}} P(\omega)$$

e.g., $P(Odd = true) = P(1) + P(3) + P(5) = 1/6 + 1/6 + 1/6 = 1/2$

Propositions

Think of a proposition as the event (set of sample points) where the proposition is true

Given Boolean random variables A and B :

event a = set of sample points where $A(\omega) = \text{true}$

event $\neg a$ = set of sample points where $A(\omega) = \text{false}$

event $a \wedge b$ = points where $A(\omega) = \text{true}$ and $B(\omega) = \text{true}$

Often in AI applications, the sample points are **defined** by the values of a set of random variables, i.e., the sample space is the Cartesian product of the ranges of the variables

With Boolean variables, sample point = propositional logic model

e.g., $A = \text{true}$, $B = \text{false}$, or $a \wedge \neg b$.

Proposition = disjunction of atomic events in which it is true

e.g., $(a \vee b) \equiv (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b)$

$\Rightarrow P(a \vee b) = P(\neg a \wedge b) + P(a \wedge \neg b) + P(a \wedge b)$

Syntax for propositions

Propositional or Boolean random variables

e.g., *Cavity* (do I have a cavity?)

Cavity = true is a proposition, also written *cavity*

Discrete random variables (*finite* or *infinite*)

e.g., *Weather* is one of $\langle \textit{sunny}, \textit{rain}, \textit{cloudy}, \textit{snow} \rangle$

Weather = rain is a proposition

Values must be exhaustive and mutually exclusive

Continuous random variables (*bounded* or *unbounded*)

e.g., *Temp = 21.6*; also allow, e.g., *Temp < 22.0*.

Arbitrary Boolean combinations of basic propositions

Prior probability

Prior or unconditional probabilities of propositions

e.g., $P(Cavity = true) = 0.1$ and $P(Weather = sunny) = 0.72$
correspond to belief prior to arrival of any (new) evidence

Probability distribution gives values for all possible assignments:

$$\mathbf{P}(Weather) = \langle 0.72, 0.1, 0.08, 0.1 \rangle \text{ (normalized, i.e., sums to 1)}$$

Joint probability distribution for a set of r.v.s gives the probability of every atomic event on those r.v.s (i.e., every sample point)

$\mathbf{P}(Weather, Cavity)$ = a 4×2 matrix of values:

$Weather =$	$sunny$	$rain$	$cloudy$	$snow$
$Cavity = true$	0.144	0.02	0.016	0.02
$Cavity = false$	0.576	0.08	0.064	0.08

Every question about a domain can be answered by the joint distribution because every event is a sum of sample points

Conditional probability

Conditional or posterior probabilities

e.g., $P(\text{cavity}|\text{toothache}) = 0.8$

i.e., **given that toothache is all I know**

NOT “if *toothache* then 80% chance of *cavity*”

(Notation for conditional distributions:

$\mathbf{P}(\text{Cavity}|\text{Toothache}) = 2\text{-element vector of 2-element vectors})$

If we know more, e.g., *cavity* is also given, then we have

$P(\text{cavity}|\text{toothache}, \text{cavity}) = 1$

Note: the less specific belief **remains valid** after more evidence arrives, but is not always **useful**

New evidence may be irrelevant, allowing simplification, e.g.,

$P(\text{cavity}|\text{toothache}, \text{49ersWin}) = P(\text{cavity}|\text{toothache}) = 0.8$

This kind of inference, sanctioned by domain knowledge, is crucial

Conditional probability

Definition of conditional probability:

$$P(a|b) = \frac{P(a \wedge b)}{P(b)} \text{ if } P(b) \neq 0$$

Product rule gives an alternative formulation:

$$P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$$

A general version holds for whole distributions, e.g.,

$$\mathbf{P}(Weather, Cavity) = \mathbf{P}(Weather|Cavity)\mathbf{P}(Cavity)$$

(View as a 4×2 set of equations, **not** matrix mult.)

Inference by enumeration

Start with the joint distribution:

	<i>toothache</i>		\neg <i>toothache</i>	
	<i>catch</i>	\neg <i>catch</i>	<i>catch</i>	\neg <i>catch</i>
<i>cavity</i>	.108	.012	.072	.008
\neg <i>cavity</i>	.016	.064	.144	.576

For any proposition ϕ , sum the atomic events where it is true:

$$P(\phi) = \sum_{\omega: \omega \models \phi} P(\omega)$$

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$$P(\text{toothache}) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$$

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$$P(\text{cavity} \vee \text{toothache}) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$$

Inference by enumeration

Start with the joint distribution:

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	<i>catch</i>	\neg <i>catch</i>	<i>catch</i>	\neg <i>catch</i>
<i>cavity</i>	.108	.012	.072	.008
\neg <i>cavity</i>	.016	.064	.144	.576

Can also compute conditional probabilities:

$$\begin{aligned}
 P(\neg \text{cavity} | \text{toothache}) &= \frac{P(\neg \text{cavity} \wedge \text{toothache})}{P(\text{toothache})} \\
 &= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4
 \end{aligned}$$

Normalization

	<i>toothache</i>		\neg <i>toothache</i>	
	<i>catch</i>	\neg <i>catch</i>	<i>catch</i>	\neg <i>catch</i>
<i>cavity</i>	.108	.012	.072	.008
\neg <i>cavity</i>	.016	.064	.144	.576

Denominator can be viewed as a normalization constant α

$$\begin{aligned}
 \mathbf{P}(Cavity|toothache) &= \alpha \mathbf{P}(Cavity, toothache) \\
 &= \alpha [\mathbf{P}(Cavity, toothache, catch) + \mathbf{P}(Cavity, toothache, \neg catch)] \\
 &= \alpha [\langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle] \\
 &= \alpha \langle 0.12, 0.08 \rangle = \langle 0.6, 0.4 \rangle
 \end{aligned}$$

General idea: compute distribution on query variable
by fixing **evidence variables** and summing over **hidden variables**

Inference by enumeration, contd.

Let \mathbf{X} be all the variables. Typically, we want the posterior joint distribution of the query variables \mathbf{Y} given specific values \mathbf{e} for the evidence variables \mathbf{E}

Let the hidden variables be $\mathbf{H} = \mathbf{X} - \mathbf{Y} - \mathbf{E}$

Then the required summation of joint entries is done by summing out the hidden variables:

$$\mathbf{P}(\mathbf{Y}|\mathbf{E} = \mathbf{e}) = \alpha \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}) = \alpha \sum_{\mathbf{h}} \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}, \mathbf{H} = \mathbf{h})$$

The terms in the summation are joint entries because \mathbf{Y} , \mathbf{E} , and \mathbf{H} together exhaust the set of random variables

Obvious problems:

- 1) Worst-case time complexity $O(d^n)$ where d is the largest arity
- 2) Space complexity $O(d^n)$ to store the joint distribution
- 3) How to find the numbers for $O(d^n)$ entries???